Math 564: Advance Analysis 1
Lecture 15

Thm. If $f_n \to_\mu f$ in measure (e.g. when $f_n \to_{L^1} f$), there is a subsequence (f_{n_k}) that converges to f a.e.

Observe (at home) that if $f_n \to_\mu f$ then (f_n) is Cauchy in measure, so it is enough to prove the following stronger theorem:

Theorem (Completeness of convergence in measure). Let (f_n) be Cauchy in measure. Then $\exists$ $\mathcal{S}$-measurable function f s.t.

(a) $f_{n_k} \to f$ a.e. for some subsequence (f_{n_k})

(b) $f_n \to_\mu f$.

Proof. For (b), it's enough to show $f_{n_k} \to_\mu f$ for some subsequence, by HW. In other words, we can, WLOG, restrict to subsequences.

Claim (Speeding up). WLOG, up to restricting to a subsequence, we may assume that $\delta_{2^{-n}}(f_n, f_{n+m}) < 2^{-n}$.

Proof. We build a subsequence (f_{n_k}) s.t. $\forall k$ $\delta_{2^{-k}}(f_{n_k}, f_{n_{k+m}}) < 2^{-k}$.

Let n_0 be s.t. $\forall n \geq n_0$ we have $\delta_{2^{-0}}(f_{n_0}, f_n) < 2^{-0}$.

Let $n_1 > n_0$ be s.t. $\forall n \geq n_1$, we have $\delta_{2^{-1}}(f_{n_1}, f_n) < 2^{-1}$.

$\vdots$

Let $n_k > n_{k-1}$ - - - $\forall n \geq n_k$, - — . $\delta_{2^{-k}}(f_{n_k}, f_n) < 2^{-k}$.

Then (f_{n_k}) is as desired. $\boxed{\text{Claim}}$

Thus, the measures of the sets $\Delta_{2^{-n}}(f_n, f_{n+1})$ are $< 2^{-n}$, and hence are summable. By Borel–Cantelli, for a.e. x $\exists N_x$ s.t. $x \notin B_{N_x} := \bigcup_{n \geq N_x} \Delta_{2^{-n}}(f_n, f_{n+1})$. We show that $(f_n(x))$ is Cauchy. Indeed, $\forall n \geq N_x$

$$|f_n(x) - f_{n+m}(x)| \leq \sum_{i=0}^{m-1} |f_{n+i}(x) - f_{n+i+1}(x)| \leq \sum_{i=0}^{m-1} 2^{-(n+i)} < 2^{-n+1}. \qquad (*)$$

Thus, $f_n(x) \to$ some number denoted by $f(x)$. f is a ptwise limit measurable functions, and hence is measurable.

We also show that $f_n \to_\mu f$. Fix $\alpha > 0$ and let N be s.t. $2^{-N+1} < \alpha$. Need to show $\mu(\Delta_\alpha(f_n, f)) \to 0$ as $n \to \infty$.

But (*) implies that $\forall x \notin B_N$, and $\forall n \geq N$, $|f_n(x) - f(x)| \leq 2^{-n+1}$

So $\forall n \geq N$, $\Delta_\alpha(f_n, f) \subseteq \Delta_{2^{-N+1}}(f_n, f) \subseteq B_N$, so $\delta_\alpha(f_n, f) \leq \mu(B_N) \leq$
$\leq \sum_{n \geq N} \mu(\Delta_{2^{-n}}(f_n, f_{n+1})) \leq \sum_{n \geq N} 2^{-n} = 2^{-N+1}$, which is arbitrarily small. □

Uniform vs. ptwise convergence.

Swapping quantifiers trick. Let (X, μ) be a finite measure space and let (P_n) be an increasing seq. of measurable sets, think of $x \in P_n$ as x satisfies P_n. If $\forall^{100\%}_x \in X\ \exists n \in \mathbb{N}\ P_n(x)$ holds, then $\exists n\ \forall^{99\%}_x \in X\ P_n(x)$ holds.

$\{x \in X : P_n(x) \text{ fails}\}$ has measure $\leq \varepsilon$.

Proof. Observe that $X = \bigcup_n^{\uparrow} P_n$ so $\mu(P_n) \to \mu(X) < \infty$, thus $\mu(X \setminus P_n) \to 0$. □

99% uniform convergence (Egorov's theorem). Let (X, μ) be a finite measure space. If $f_n \to f$ a.e. then $\forall \delta > 0\ \exists X' \subseteq X$ measurable $\mu(X \setminus X') \leq \delta$ on which $f_n \to f$ uniformly.

Proof. We have $\forall \varepsilon > 0\ \forall^{100\%}_x \in X\ \exists N$ s.t. $\overbrace{|f_n(x) - f(x)| < \varepsilon\ \forall n \geq N}^{P_N}$

Thus $\forall \varepsilon > 0 \quad \exists N_\varepsilon \quad \forall^{99\%} x \in X \quad P_{N_\varepsilon}(x)$ holds.

Let $\varepsilon_k \to 0$ (e.g. $\varepsilon_k := \frac{1}{k}$). Then

$\forall \varepsilon_k \quad \exists N_k \quad \forall^{-\delta_k} x \in X \quad P_{N_k}(x)$ holds, where $\delta_k := \delta \cdot 2^{-k-1}$.

In other words, the set

$$X_k := \{ x \in X : |f_n(x) - f(x)| < \varepsilon_k \ \forall n \geq N_k \}$$

has measure $\geq \mu(X) - \delta_k$.

Then $X' := \bigcap_{k=0}^{\infty} X_k$ works. $\mu(X \setminus X') \leq \sum_{k \in \mathbb{N}} \mu(X \setminus X_k) \leq \sum \delta \cdot 2^{-k-1} \leq \delta$.

For all $\varepsilon_k \quad \forall x \in X' \quad |f_n(x) - f(x)| < \varepsilon_k \quad \forall n \geq N_k$. $\square$

Product measures and Fubini.

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Let $\mathcal{M} \otimes \mathcal{N}$ denote the σ-algebra generated by rectanges, i.e. sets of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

Prop. There is a measure ρ on $\mathcal{M} \otimes \mathcal{N}$ s.t. $\rho(A \times B) = \mu(A) \cdot \nu(B)$ for all $A \times B \in \mathcal{M} \times \mathcal{N}$. This ρ is unique if μ and ν are σ-finite, in which case we denote it by $\mu \times \nu$.

Proof. Let $\mathcal{A}$ be the algebra generated by rectanges, hence $\mathcal{A} = \{$finite unions of rectangles$\}$. We define the measure ρ on $\mathcal{A}$ by

$$\rho\left(\bigsqcup_{n<N} A_n \times B_n \right) := \sum_{n<N} \mu(A_n) \cdot \mu(B_n).$$

To show that ρ is well-defined and ctbl-additive on $\mathcal{A}$, it is enough to show that if a rectangle $A \times B$ is equal to $\bigsqcup_{n \in \mathbb{N}} A_n \times B_n$ then

$$\mu(A) \cdot \nu(B) = \sum_{n \in \mathbb{N}} \mu(A_n) \cdot \nu(B_n).$$

We show this using integrals. We have that $\mathbb{1}_{A\times B} = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n\times B_n}$, and $\mathbb{1}_{A\times B}(x,y) = \mathbb{1}_A(x)\cdot\mathbb{1}_B(y)$. We integrate both sides wrt y and get, for each $x\in X$:

$$\begin{array}{ccc} \displaystyle\int_Y \mathbb{1}_A(x)\cdot\mathbb{1}_B(y)\,d\nu(y) = \int_Y \sum_n \mathbb{1}_{A_n}(x)\,\mathbb{1}_{B_n}(y)\,d\nu(y) \overset{MCT}{=} \sum_n \int_Y \bigcirc\, d\nu \\ \quad\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \mathbb{1}_A(x)\cdot\nu(B) \qquad\qquad\qquad\qquad \displaystyle\sum_n \mathbb{1}_{A_n}(x)\cdot\nu(B_n). \end{array}$$

We integrate again over x and again use MCT:

$$\nu(B)\cdot\mu(A) = \nu(B)\cdot\int_X \mathbb{1}_A(x)\,d\mu(x) = \sum_n \nu(B_n)\cdot\int_X \mathbb{1}_{A_n}(x)\,d\mu(x) = \sum_n \nu(B_n)\cdot\mu(A_n).$$

Thus, ρ is a premeasure on $\mathcal{A}$, hence Carathéodory's theorem applies. □

<u>Fubini-Tonelli theorem.</u> Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces. Let f be a $\mu\times\nu$-measurable function $X\times Y\to[-\infty,\infty]$

(a) Tonelli. If $f\geq 0$, then for μ-a.e. $x\in X$, $f_x : Y\to[0,\infty]$ ν-measurable and for ν-a.e. $y\in Y$, $f_y : X\to[0,\infty]$ is μ-measurable, and the functions $g: X\to[0,\infty]$, $x\mapsto\int f_x\,d\nu$ and $h: Y\to[0,\infty]$, $y\mapsto\int f_y\,d\mu$ are both measurable, and

$$\int_Y \overbrace{\int_X f\,d\mu}^{h}\,d\nu = \int_{X\times Y} f\,d\mu\times\nu = \int_X \overbrace{\int_Y f\,d\nu}^{g}\,d\mu.$$

(b) Fubini. If $f\in L^1(X\times Y,\mu\times\nu)$, then same holds, but in addition, $g\in L^1(X,\mu)$ and $h\in L^1(Y,\nu)$.